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Common fixed point theorems for the pair of mappings in Hilbert space

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ABSTRACT

In this paper common fixed point theorem for the pair of mapping satisfying different contractive condition in Hilbert space has been proved.

Key words: Fixed point, Hilbert space, contraction mapping, Banach space.

INTRODUCTION

Most of fixed point theorems for mappings in metric spaces satisfying different contraction conditions may be extended to the abstract spaces, like Hilbert, Banach and locally convex spaces etc with some modifications. Some such interesting classes of contraction by Ciric[1],Dotson[2] proved fixed point theorems for non-expansive mappings on star shaped subsets of Banach spaces (i.e. $\|Tx-Ty\| \le \|x-y\|$ for $x,y \in C$). Then T has a fixed point in C. Pandhare and Waghmode[3] have proved class of pairs of generalized contraction type mapping in Hilbert space on the line of Ciric[1] and proved some common fixed point theorems and some such interesting classes of contraction introduced by Kannan[4]. Sayyed and Badshah[5] proved a class of pair of generalized contraction type mapping in Hilbert space. The result of this theorem is inspired by the results due to Dubey[6], Naimpally and Singh[7].

Definition

Let X be a Banach space and C be a non-empty subset of X. Let $T_1, T_2 : C \to C$ be two mappings. The iteration scheme called I-scheme is defined as follows :

$$x_0 \in C, \quad \text{...(1)}$$

$$y_{2n} = \beta_{2n}T_1x_{2n} + (1-\beta_{2n})x_{2n},\ n \ge 0$$

$$x_{2n+1} = (1-\alpha_{2n})x_{2n} + \alpha_{2n}T_2y_{2n},\ n \ge 0 \quad \text{...(2)}$$

$$y_{2n+1} = \beta_{2n+1}T_1x_{2n+1} + (1-\beta_{2n+1})x_{2n+1},\ n \ge 0$$

$$x_{2n+2} = (1-\alpha_{2n+1})x_{2n+1} + \alpha_{2n+1}T_2y_{2n+1},\ n \ge 0 \quad \text{...(3)}$$

In the Ishikawa scheme, $\{\alpha_{2n}\}$, $\{\beta_{2n}\}$ satisfy $0 \le \alpha_{2n} \le \beta_{2n} \le 1$, for all n $\lim_{n\to\infty} . \beta_{2n} = 0$ and $\Sigma\alpha_{2n}\beta_{2n} = \infty$. In this paper we shall make the assumption that

(i) $0 \le \alpha_{2n} \le \beta_{2n} \le 1$, for all n,

(ii) $\lim_{n\to\infty} \alpha_{2n} = \alpha_{2n} > 0$, and

(iii) $\lim_{n\to\infty} \beta_{2n} = \beta_{2n} < 1$.

We know that Banach space is Hilbert if and only if its norm satisfies the parallelogram law i.e. every $x,y \in X$ (Hilbert space).

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \text{...(4)}$$

which implies, $\|x + y\|^2 \le 2\|x\|^2 + 2\|y\|^2$ ...(5)

We often use this inequality throughout the result.

Further, we prove the result concerning the existence of common fixed point of pairs of mappings satisfying the contraction condition of the type

$$||Tx-Ty||^2 \le h \text{ Max } \{ ||x - y||^2, ||x - Tx||^2, ||y - Ty||^2, 1/4(||x-Ty||^2 + ||y - Tx||^2)\} \quad ...(6)$$

Theorem

Let X be a Hilbert space and C be a closed, convex subset of X. Let T_1 and T_2 be two sets of mapping satisfying

$$||T_1x-T_2y||^2 \le h\text{Max}\{||x - y||^2, ||x - T_1x||^2, ||y-T_2y||^2, 1/4(||x-T_2y||^2+||y-T_1x||^2)\} \quad ...(7)$$

where h is real number satisfying $0 \le h < 1$. If there exists a point x_0 such that the I-scheme for T_1 and T_2 defined by (2) and (3) converges to a point p, then p is common fixed point of T_1 and T_2.

Proof

It follows from (2) that $x_{2n+1} - x_{2n} = \alpha_{2n}(T_2y_{2n} - x_{2n})$. Since $x_{2n} \to p$, $||x_{2n+1} - x_{2n}|| \to 0$. Since $\{\alpha_{2n}\}$ is bounded away from zero, $||T_2y_{2n} - x_{2n}|| \to 0$. It also follows that $||p - T_2y_n|| \to 0$. Since T_1 and T_2 satisfies (7), we have

$$||T_1x_{2n}-T_2y_{2n}||^2 \le h\text{Max}\{||x_{2n}-y_{2n}||^2, ||x_{2n}-T_1x_{2n}||^2, ||y_{2n} - T_2y_{2n}||^2, 1/4\ (||x_{2n}-T_2y_{2n}||^2+||y_{2n}-T_1x_{2n}||^2)\} \quad ...(8)$$

Now, $||y_{2n}- x_{2n}||^2 = ||\beta_{2n}T_1x_{2n} + (1- \beta_{2n})\ x_{2n}- x_{2n}||^2 = ||\beta_{2n}T_1x_{2n}+x_{2n} - \beta_{2n}x_{2n}- x_{2n}||^2$

$$= ||\beta_{2n}(T_1x_{2n} - x_{2n})\ ||^2 = \beta_{2n}{}^2 ||\ (T_1x_{2n} - T_2y_{2n}) + (T_2y_{2n} - x_{2n})\ ||^2 \le 2\beta_{2n}{}^2||T_1x_{2n}-T_2y_{2n}||^2+2\beta_{2n}{}^2||(T_2y_{2n}-x_{2n})||^2 \le 2||T_1x_{2n}-T_2y_{2n}||^2+2||(T_2y_{2n}-x_{2n})||^2 \quad ...(9)$$

$$||y_{2n} - T_2y_{2n}||^2 = ||\beta_{2n}T_1x_{2n} + (1- \beta_{2n})\ x_{2n} - T_2y_{2n}||^2$$
$$=||\beta_{2n}T_1x_{2n}+(1-\beta_{2n})x_{2n}-T_2y_{2n}+\beta_{2n}T_2y_{2n}-\beta_{2n}T_2y_{2n}||^2$$
$$=||\beta_{2n}(T_1x_{2n} - T_2y_{2n}) + (1- \beta_{2n})\ (x_{2n} - T_2y_{2n})\ ||^2$$
$$\le 2\beta_{2n}{}^2\ ||T_1x_{2n} - T_2y_{2n}||^2 + 2(1- \beta_{2n})\ ||\ x_{2n} - T_2y_{2n}||^2$$
$$\le 2||T_1x_{2n} - T_2y_{2n}||^2 + 2||x_{2n} - T_2y_{2n}||^2 \quad ...(10)$$

$$||y_{2n} - T_1x_{2n}||^2 = ||\beta_{2n}T_1x_{2n} + (1- \beta_{2n})\ x_{2n} - T_1x_{2n}||^2 = ||(1- \beta_{2n})(x_{2n} - T_1x_{2n})\ ||^2$$
$$= (1- \beta_{2n})^2\ ||x_{2n} - T_1x_{2n}||^2 = (1-\beta_{2n})^2\ ||(x_{2n} - T_2y_{2n}) + (T_2y_{2n} - T_1x_{2n})||^2$$
$$\le 2(1-\beta_{2n})^2||x_{2n}-T_2y_{2n}||^2+2(1-\beta_{2n})^2||T_2y_{2n}-T_1x_{2n}||^2$$
$$\le 2||x_{2n}-T_2y_{2n}||^2 + 2\ ||T_2y_{2n} - T_1x_{2n}||^2 \quad ...(11)$$

from (8), (9), (10) and (11) can be written as :

$$||T_1x_{2n}-T_2y_{2n}||^2 \le h\max\{2||T_1x_{2n} - T_2y_{2n}||^2 + 2\ ||T_2y_{2n} - x_{2n}||^2),$$
$$2||x_{2n} - T_2y_{2n}||^2 + 2\ ||T_2y_{2n} - T_1x_{2n}||^2,\ 2||T_1x_{2n} - T_2y_{2n}||^2 + 2||x_{2n} - T_2y_{2n}||^2),\ 1/4(3||x_{2n}-T_2y_{2n}||^2+2||T_2y_{2n}-T_1x_{2n}||^2,)\} \le h\ (2||T_1x_{2n} - T_2y_{2n}||^2 + 2||T_2y_{2n} - x_{2n}||^2)$$
$$\le 2h/1\text{-}2h\ ||x_{2n} - T_2y_{2n}||^2$$

Taking limit as $n \to \infty$, we get $||T_1x_{2n} - T_2y_{2n}|| \to 0$. It follows that $||x_{2n} - T_1x_{2n}||^2 \le 2||x_{2n} - T_2y_{2n}||^2 + 2||T_2y_{2n} - T_1x_{2n}||^2 \to 0$.

And $||p - T_1x_{2n}||^2 \le 2||p - x_{2n}||^2 + 2||x_{2n} - T_1y_{2n}||^2 \to 0$ as $n \to \infty$.

If x_{2n}, p satisfies (7), we have

$$||T_1x_{2n}-T_2p||^2 \le h\max\{||x_{2n} - p||^2, ||x_{2n} - T_1x_{2n}||^2, ||p - T_2p||^2, 1/4(||x_{2n} - T_2p||^2 + ||p - T_1x_{2n}||^2)\}$$
$$\le h\max\{||x_{2n} - p||^2, ||x_{2n} - T_1x_{2n}||^2, ||p - x_{2n} + x_{2n} - T_2p||^2, 1/4(||x_{2n}-T_1x_{2n}+T_1x_{2n}-T_2p||^2+||p - T_1x_{2n}||^2)\}$$

Using inequality (5), we have

$$||T_1x_{2n}-T_2p||^2 \le h\max\{||x_{2n} - p||^2, ||x_{2n} - T_1x_{2n}||^2, 2||x_{2n} - p||^2, + 4||x_{2n}-T_1x_{2n}||^2 + 4||T_1x_{2n} - T_2p||^2, 1/4(2||x_{2n} - T_1x_{2n}||^2 + 2||T_1x_{2n} - T_2p||^2 + ||p - T_1x_{2n}||^2)\}$$

Taking limit as $n \to \infty$, we get $||T_1x_{2n} - T_2p|| \to 0$.
Finally, $||p - T_2p||^2 = ||p - T_1x_{2n} + T_1x_{2n} - T_2p||^2 \le 2||p - T_1x_{2n}||^2 + 2||T_1x_{2n} - T_2p||^2 \to 0$, as $n \to \infty$.

Showing that $p = T_2\,p$. Similarly, we can prove that $p = T_1\,p$. Thus p is a common fixed point of T_1 and T_2. This completes the proof.

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